Solutions to Problems

General Relativity

The Postulates of General Relativity

1.1 It was said that to an observer in freefall there is no gravity. Let's check out one case. Suppose at some time Ophelia is in freefall, and she is traveling downward at 3 m/s. At this time, a ball is launched upwards at 15 m/s.

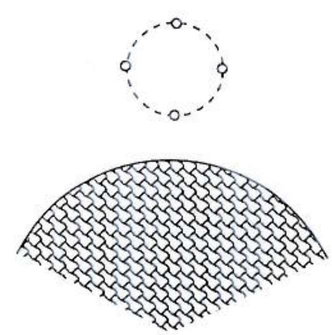
a) What is the speed of the ball relative to Ophelia?

b) 1 second later, what are the speeds of Ophelia, the ball, and the ball relative to Ophelia?

c) What will the speed of the ball be relative to Ophelia for other times?

1.2

Imagine a group of four marbles spread out on an imaginary circle as shown. If they are released from rest and allowed to fall freely, how will they move relative to each other? (Consider only the force of gravity from the planet acting on each marble; ignore their mutual attraction.)



1.3 Design a device using springs and masses to measure tidal effects inside your spaceship. (It could be used as an alarm in case you get too close to a black hole.)

1.4 What will remain of gravity if you fall freely in a gravitational field?

1.5 Does general covariance mean that the forces you feel will be the same in all frames of reference?

1.6 You get the opportunity to solve a projectile problem in the 'no gravity' (freefall) frame of reference. Imagine a projectile being launched from the floor ($y_i = 0$) with a speed of 5 m/s at an angle of 60°. If you are falling freely, it will follow a straight line path relative to you. As a result, its equations of motion are

$x = v_x t$ and $y = v_y t$, where $v_x = 5\cos 60°$ and $v_y = 5\sin 60°$.

The floor is at the same height as the ball initially ($y_{i\ floor} = 0$), but the floor is accelerating upward. (Remember, gravity down is the same as accelerating the room upward.) So the equation of motion for the floor is

$y_{floor} = (½)gt^2$.

a) At what time will the ball and the floor be at the same y coordinate?

b) What will the x coordinate of the ball be at that time?

You have just calculated where the ball will hit the ground. Congratulations!

1.7 Suppose you are in your car and you accelerate forward. Your fuzzy dice hang backward at an angle of 20°. What is your acceleration? (Hint: Think like Einstein.)

1.8 Why do fictitious forces appear for non-inertial observers?

Non-Euclidean Geometry

2.1 What type of world did O'Malley find himself in? How about Ophelia? Can you draw a map of their travels?

2.2 Why are most lines of latitude not straight lines? Can you find a clear example of a line of latitude which is not the shortest distance between two points?

2.3 For two lines on a sphere, how many points of intersection are possible?

2.4 On a plane, if you have a line which crosses one line of a pair of non-intersecting lines, it must cross the other as well. (We call that line a transversal.) Is this still true on curved surfaces? (The answer may differ for spheres and saddles.)

2.5 What happens to the curvature of a balloon as you inflate it?

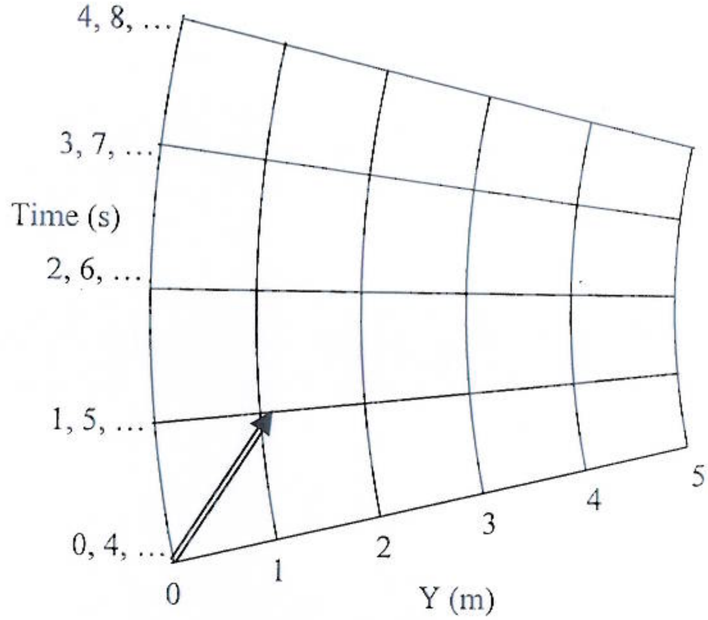
2.6 Why is the curvature of a cylinder zero?

2.7 How many straight lines connect two points on a cylinder?

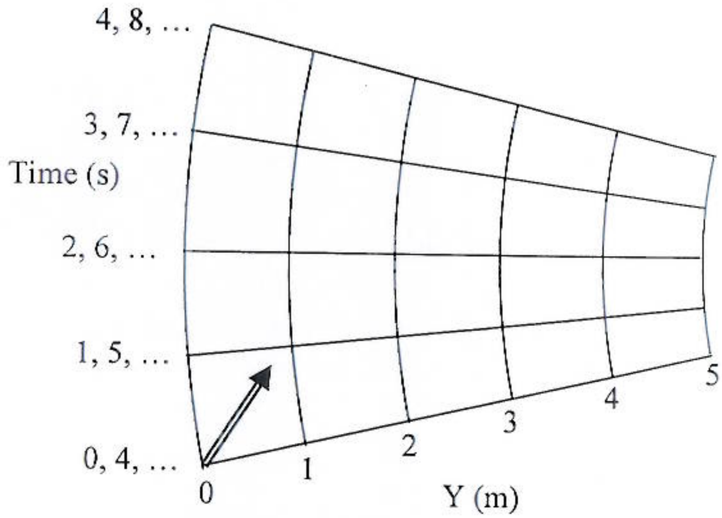
2.8 Where is the curvature of the surface of a torus (doughnut, inner tube) positive, and where is it negative?

Gravity and Projectile Motion

3.1 Try drawing a trajectory for yourself on two graphs with different amounts of curvature. For each graph find the maximum height of the projectile. Which graph shows stronger gravity? Which has more curvature (is more distorted from a rectangle)? (The initial speed of the projectile is related to the angle the arrow makes with the axes. In each of the graphs below, the arrow makes the same initial angle.)



A

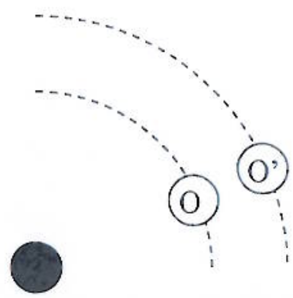


B

3.2 Qualitatively, how does the maximum height of the projectile depend on the amount of curvature? Explain. What would the maximum height of a projectile be if the curvature were infinite?

3.3 Is it possible to "jump off the surface" of the deformed spandex and take a shortcut getting from one point to another?

3.4 Two explorers in spaceships are orbiting a black hole. The first, Ophelia, calculated her radius to be 100,000 meters, and the second, O'Malley, finds that his radius is 150,000 meters. At a time when they are on the same side of the black hole, as shown, O'Malley puts out a tape measure to directly measure their separation. Does he find it to be 50,000 meters, more than 50,000 meters, or less than 50,000 meters? Explain.



what would its trajectory look like to you?

3.5 If you were in a free-falling elevator and you used a slingshot to shoot a pea across the elevator, ~~would it follow a straight path~~ (as viewed by you)? ~~What if you shot a pulse of light?~~ Please explain.

3.6 This problem points out another problem with Newton's view of gravity.

Suppose you are in space, and you've got two asteroids pulling on each other. By knowing the distance between their centers and their masses, you can calculate the force of each pulling on the other. Imagine that you have rocket engines pushing outward on each asteroid to keep them at the same distance. The rockets have force measuring devices so they can show just how much force they are providing. Calculating the value of the force from Newton's Law of Universal Gravitation gives a value in agreement with the rocket measurements. No surprise there.

Next, suppose another observer shoots by at close to the speed of light.

a) What will this observer say about the distance between the two asteroids compared with the first astronaut?

b) What will this observer say about the two masses compared with the first astronaut?

c) When this observer calculates the force between the asteroids (using Newton's Gravity) will the answer agree or disagree with the rocket measurements?

Three Classical Tests

Perihelion Problems

4.1 Why would Mercury show the largest perihelion shift (of all the planets) due to General Relativity?

Bending of Light

4.2 Suppose you have two stars in the night sky. Also suppose that six months later the Sun is directly between those two stars (because the Earth is now on the other side of the Sun). Will the two stars appear closer together or further apart now that the Sun lies between them in the sky?

4.3 Suppose you had three spacecraft surrounding a black hole. The astronaut in each ship points a laser beam so that it hits the next ship. The laser beams form a triangle of sorts. How will the sum of the angles of the triangle compare to 180 degrees?

4.4 What is the maximum bending of light as it passes by the Earth?

4.5 Suppose you wanted to use the bending of light by the Sun as a telescope lens. If a source of light was many light years away, where would the light rays passing on either side of the Sun converge?

4.6 If a massive object, such as a large planet, passes in front of a distant star, the brightness of the star can be increased temporarily. Explain how this can happen. (The effect is called microlensing.)

Gravitational Redshift

4.7 What is the fractional change of frequency ($\Delta f/f$) of light leaving the surface of the Sun and heading out to infinity? What about light leaving the surface of the Earth?

4.8 Suppose that O'Malley, at the top of the spaceship, has the laser and points it "down" toward Ophelia. What will she have to say about the frequency of the laser? Why?

4.9 Cosmological redshifts (due to the expansion of the universe) have been observed that produce $f_{observed}$ values that are as small as ¼ of f_{source}. How does the gravitational redshift from the surface of a star compare to this?

Gravitational Time Dilation

5.1 If you live 100 years on the surface of the Earth, how long would an observer at infinity claim you had lived? (You should ignore the existence of the Sun in this problem, and just calculate the effect due to the Earth's gravity.) Hint: A Maclaurin series may help you here.

5.2 To find the relationship between the periods of two clocks, both at finite locations, you can do the following: Write out the formulas comparing the period of each clock to a clock at infinity, and then use these two equations to eliminate the period of the clock at infinity. What is the resulting formula?

5.3 GPS (Global Positioning System) satellites travel in LEO (Low Earth Orbit) at an altitude of about 400 km. Compared to a clock on the surface of the Earth, by what factor will the clock be off? Will it be slow or fast relative to the Earth clock? (The effect of its speed will slow it; less gravity than on the surface of the Earth will speed it up.) Note: This effect was calculated in advance because it is big enough to keep the GPS system from working properly if not accounted for.

5.4 If you wanted to live a bit longer, should you live at the ground floor of a building or in the penthouse? For this problem, consider only the effect of gravity and ignore the velocity of each apartment.

5.5 A clock on the surface of the Earth (we'll assume it is at the equator to simplify things) runs slow both because of the presence of the Earth's mass and due to its speed which results from the Earth's rotation. A satellite orbiting the Earth is moving at a different speed ($v^2 = GM/r$, according to our work on orbits last fall) and is at a different radius. Is there an orbit in which a satellite's clock would have the same dilation as a clock on the Earth's equator? If so, how high above the surface of the Earth will it be?

Black Holes

6.1 What is the Schwarzschild radius for a black hole with the mass of the Sun (2×10^{30} kg)? Of the Earth (6×10^{24} kg)? Of the black hole at the center of our galaxy (about 2.6 million solar masses)?

6.2 Suppose a source of light is located at a distance $r = 3GM/c^2$ from a black hole. The frequency of the source is 1×10^{15} Hz, just into the ultraviolet so that it normally cannot be seen with the eye. What frequency will be observed by an observer at infinity? Will you see the light? If so, what color will you see?

6.3 Suppose a source of light is located at a distance $r = 2GM/c^2$ from the center of a black hole. The frequency of the source is 1×10^{15} Hz, just into the ultraviolet so that it normally cannot be seen with the eye. What frequency will be observed by an observer at infinity? What is the significance of this answer?

6.4 In order to jump into the future and escape the statute of limitations (it wasn't anything serious, really, just a little quantum mechanical practical joke), Phred the physics student travels to a black hole. He gets close enough to the black hole so that he will experience just one month for every year that goes by at home. If the mass of the black hole is 8×10^{30} kg, at what radius will he need to be? (You may assume he will just hover there, so that his velocity is zero.) You may answer either in meters or in multiples of the Schwarzschild radius.

6.5 Use the time dilation formula and a Maclaurin series to show that the dilation factor is just the square root of the altitude above the event horizon divided by the Schwarzschild radius, as long as the altitude is small compared to the Schwarzschild radius. (Hint: Make the radius = $r_{Schwarzschild} \cdot (1+\delta)$, where δ is very small compared to 1.)

6.6 What is the rate of evaporation (mass lost per second) for a stellar sized black hole? For the black hole at the center of our galaxy (about 3 million solar masses)?

6.7 Since a black hole evaporates faster and faster, you cannot use the current rate of evaporation to directly find the lifetime of a black hole. Use the sequence function

ability of your calculator (or a spreadsheet) to estimate the lifetime of a black hole of mass 10^6 kg. (To do this, you assume the rate is not changing too quickly over a one second period. That way, you can use the current evaporation rate to estimate the mass one second later. With this new mass, you find a new (slightly higher) evaporation rate, which you use to estimate the mass after the next second. Continue this process until you reach zero mass. For this problem, the answer should be less than 100 seconds.)

6.8 a) Use calculus to show that the formula for the lifetime of a black hole is given by

$$\text{Lifetime} = M^3 / 1.2 \times 10^{16}$$

b) Use this formula to find the lifetime of a solar mass black hole.

6.9 Here's another weird thing about black holes: Black hole thermodynamics!

a) If you add energy to a black hole, what happens to its mass?

b) What happens to the rate at which it emits Hawking radiation as a result?

c) So what has happened to the temperature of the black hole?

d) Why does this seem weird?

Cosmology

7.1 What is it like at the edge of the observable universe?

7.2 If everything in the universe were expanding at the same rate, would we know it?

7.3 What would the universe look like if instead of having space expand, atoms were shrinking?

7.4 What part(s) of the universe is (are) expanding?

7.5 Let's accept that the universe is 13.7 billion years old. If telescopes could see to a distance of 13.7 billion light years, what would they show?

7.6 Suppose that the Hubble constant were to remain constant over time. (This would be true in a universe dominated by a cosmological constant.)

a) Using the formula given for the rubber rope, $v_{\text{spot on rope}} = Hx$, show that the distance to another galaxy will grow exponentially with time. (Hint: You will need to use calculus.)

b) What does this say about the recession velocity of a given galaxy as time goes by?

7.7 Assume that the force due to the cosmological constant grows with distance. Imagine that at some distance D the force of gravity and the outward cosmological

force balance. Why is this equilibrium unstable?

7.8 Using Einstein's equation, show how the Hubble constant depends on density.

1.1

$v_{i_{ball}} = -3 \, m/s.$

$v_{i_{Ophelia}} = 15 \, m/s.$

a) The speed of the ball relative to Ophelia is

$v_{i_{ball}} - v_{i_{Ophelia}} = -3 \, m/s - 15 \, m/s = -18 \, m/s.$

b) One second later, both the ball and Ophelia will have changed speeds: $\Delta v = a \cdot \Delta t = -9.8 \cdot 1 sec.$

$\therefore v_{f_{ball}} = -3 \, m/s - 9.8 \, m/s = -12.8 \, m/s.$

Also, $v_{f_{Ophelia}} = 15 \, m/s - 9.8 \, m/s = 5.2 \, m/s.$

$\therefore$ relative speed $= v_{f_b} - v_{f_o} = -12.8 \, m/s - 5.2 \, m/s = -18 \, m/s.$

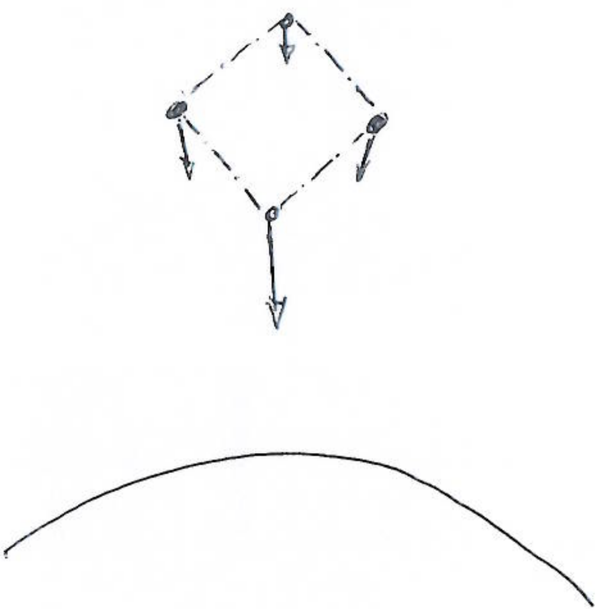
The same relative speed!

c) At any time, $v_b = 3 - 9.8t$; $v_o = 15 - 9.8t.$

$\therefore v_{relative} = v_b - v_o = (3 - 9.8t) - (15 - 9.8t) = \underline{\underline{-18 \, m/s.}}$

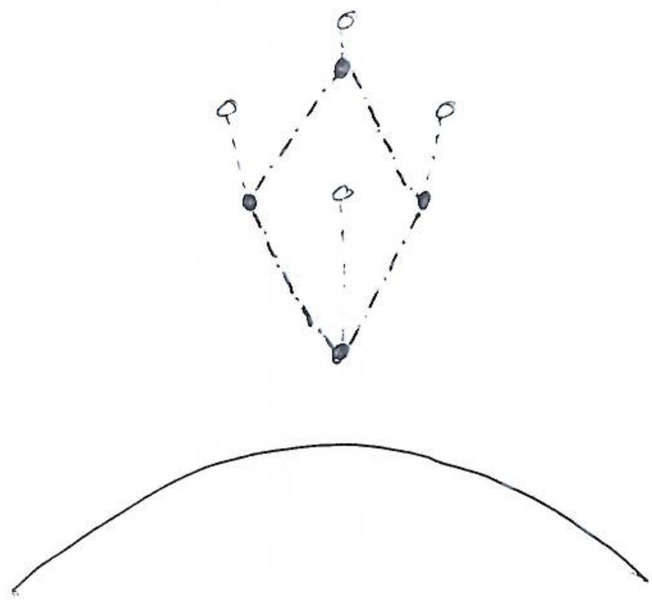
1.2

Free-body diagrams



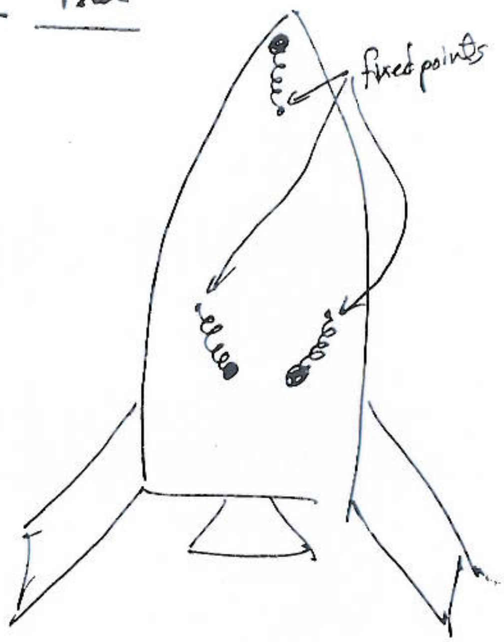
The direction of the force is toward the center of the planet. The marbles closer to the planet experience a larger force.

Resulting Motion



The marble with the greatest force travels further, stretching the pattern vertically. The left and right marbles move toward the center as well as down. Some people call this process "spaghettification."

1.3



If the springs <u>differ</u> in their extension by more than some specified value, then there is too much difference in the gravitational force across the ship for comfort.

1.4

Since the effects of gravity are locally the same as the effects of an acceleration, a person in freefall will experience no local effects of gravity.

What's left? The non-local effects, the tidal effects, are still present. So objects separated horizontally will gradually come together and objects separated vertically will separate.

Some authors say that the only real part of gravity, the stuff that all observers can agree on, is the tidal effect.

1.5

"General covariance" says that the laws of physics will be the same for all observers. But forces are not laws. In a car that accelerates forward, we feel a "force" pushing us back in our seats. In free fall, we feel no gravity. The forces we feel will depend on our frame of reference, but the laws of motion will be the same.

1.6

a) $y_{floor} = \frac{1}{2} g t^2$

$y_{projectile} = v_y \cdot t = 5 \cdot \sin 60° \cdot t$

When are they at the same height? When $y_f = y_p$.

$$y_f = y_p.$$

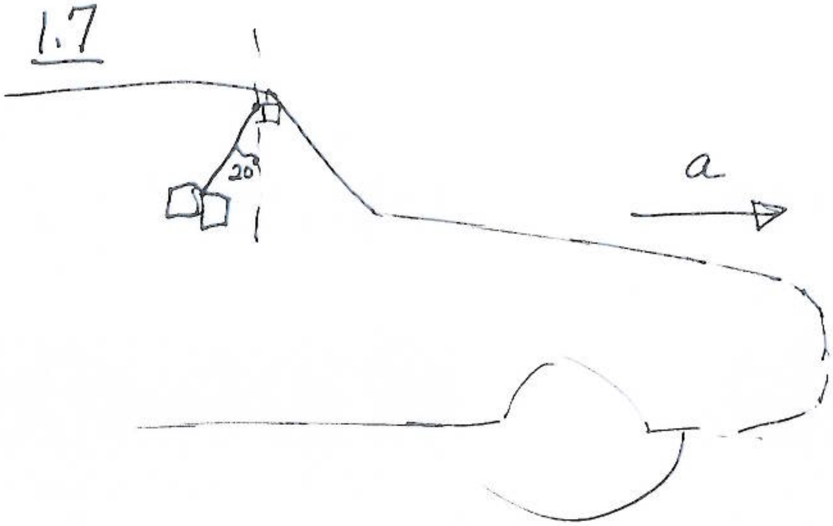
$$\frac{1}{2} g t^{\cancel{2}} = 5 \cdot \sin 60° \cdot \cancel{t}$$

$$\left[\frac{1}{2} g t = 5 \cdot \sin 60°\right] \cdot \frac{2}{g}$$

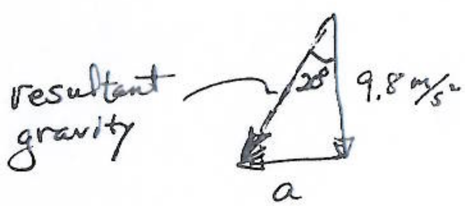
$$t = \frac{(10 \, m/s) \sin 60°}{9.8 \, m/s^2} = 0.8837 \text{ sec.}$$

b) $x = v_x t = 5 \cdot \cos 60° \cdot t = 5 \cdot \cos 60° \cdot 0.8837 = 2.209$ meters.

1.7



An acceleration to the right is like gravity to the left. So for the fuzzy dice, it's like there are two components of gravity:



From SOHCAHTOA, $\tan 20° = \frac{a}{9.8}$.

$$\therefore\ a = 9.8 * \tan 20° = 3.567\ m/s^2.$$

1.8

Suppose there is a car accelerating to the right at 5 m/s^2. If you're holding a golf ball in your hand (you are in the car) then you'll have to be pushing forward on the ball to keep it accelerating along with you.

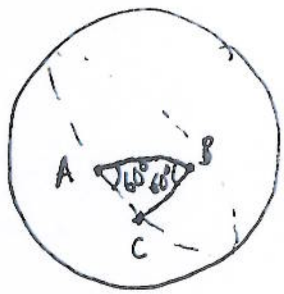
From your point of view, you are pushing forward on the ball in order to keep it at rest (relative to you). If I have to push forward to keep it still, it seems like I'm fighting some backward force. That isn't a real force — it is fictitious. (There is no backward force.)

Sidebar: How hard do I have to push?
Well, let's say the mass = 0.1 kg. And we have $a = 5\, m/s^2$. Since $F = ma$, the required force = $0.1\text{ kg} * 5\, m/s^2 = 0.5$ Newtons.

Since fictitious forces are due to acceleration, and since $F = ma$, <u>the fictitious force is always proportional to the mass of the object.</u> (Notice that <u>gravity</u> is proportional to mass.)

2.1

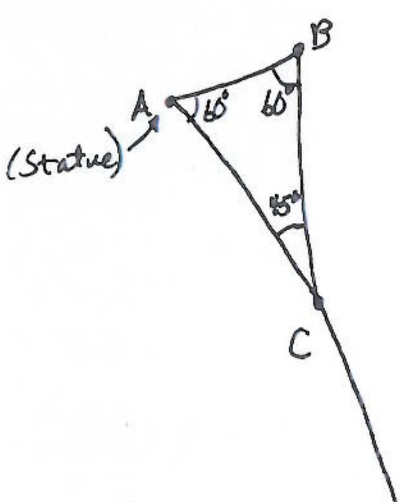
O'Malley



O'Malley traveled around a sphere, repeatedly returning to point A. Then he made a 60° angle, traveled to B, created another 60° angle, and arrived at C earlier than he expected. And the angle at C turned out to be 70° (a bit exaggerated in my sketch).

Since the sum of the angles exceeds 180°, this is a positively curved surface.

Ophelia

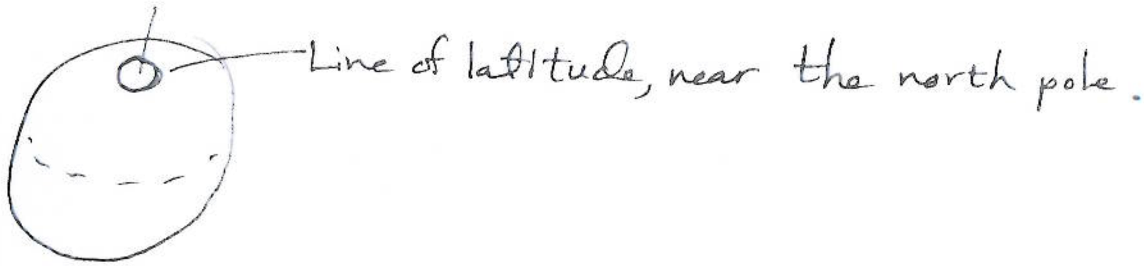


Ophelia created a triangle for which the sum of the angles was less than 180°. This is a negatively curved surface, like that of a saddle.

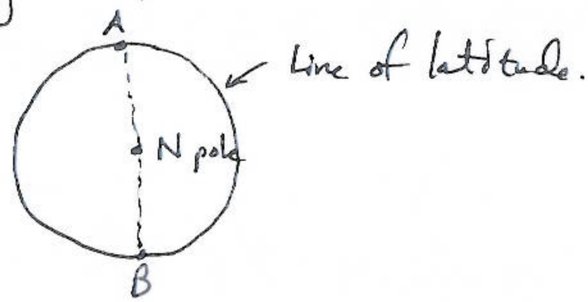
Bad pun

The rock monster could only think when standing. It stood

2.2



If we pick a line of latitude very near the pole, it will look like this (from above the pole, looking down):

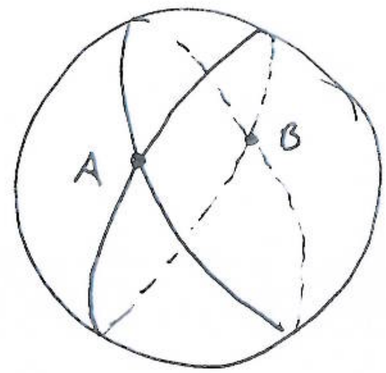


I can travel from A to B following the dashed line and travel a much shorter distance than by following the line of latitude.

∴ The line of latitude is not the shortest path.

∴ The line of latitude is not a straight line.

2.3



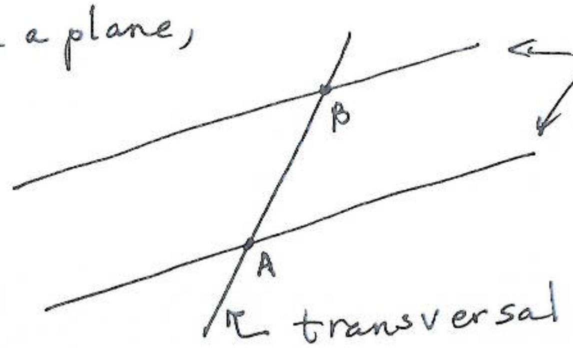
Lines on a sphere are great circles. Just like two lines of longitude that meet at the north and south poles, these two lines will meet at two points, A and B in this diagram, that are exactly opposite each other (called antipodes).

In fact, for <u>any</u> two great circles, there will always be two (and only two) points of intersection. No other answer is possible.

(On a plane, two lines will either have 0 or 1 intersection points — 0 for the special case of parallel lines, 1 otherwise.)

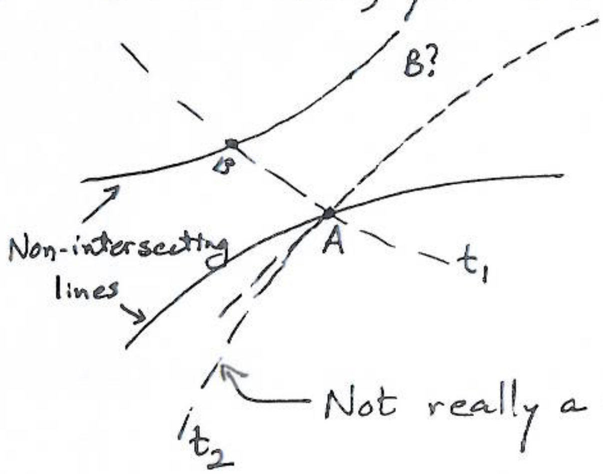
2.4

On a plane,



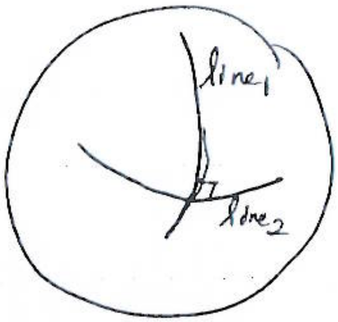
On a sphere, you can't even begin this problem because you can't have a pair of non-intersecting lines. (Any pair of lines on a sphere will intersect.)

On a saddle, you can have non-intersecting lines. They have a place where they are closest, but unlike parallel lines on a plane their distance of separation varies.



Line t_1 crosses both lines, but t_2 only intersects one of the non-intersecting lines.

2.#5



Straight lines on a sphere are great circles, and their radius is just the radius of the sphere.

$\therefore$ radius of line 1 = radius of line 2 = r.

$\therefore$ Curvature $= \dfrac{1}{r_1 \cdot r_2} = \dfrac{1}{r^2}$, for a sphere.

As r gets larger, the curvature gets lower.

When you think about this, it makes some sense: if you look at the surface of a really big sphere, it looks nearly flat. The bigger the sphere, the closer to flat it becomes.

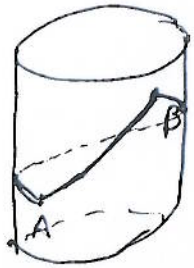
2.6

$$\text{Curvature} = \frac{1}{r_a * r_b}.$$

If r_a is the radius of the circle formed by line a, r_a = radius of the cylinder. But line b is a truly straight line, which can be considered an arc of a circle of infinite radius. $\therefore r_b = \infty$.

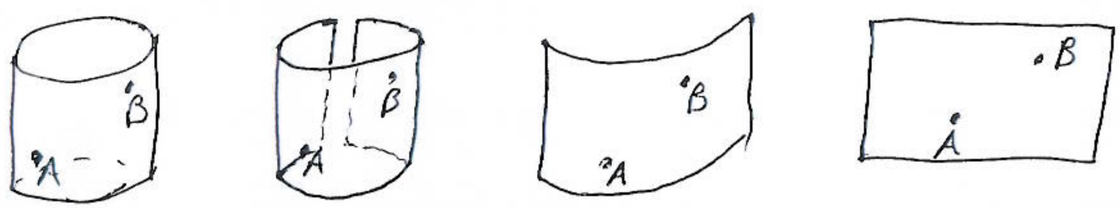
$$\text{Curvature} = \frac{1}{r_a \cdot \infty} = \frac{1}{\infty} = 0.$$

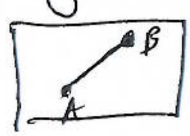
2.7



Here are two straight lines that connect A to B.

One way to look at a cylinder is to cut it vertically along one line and flatten it out:

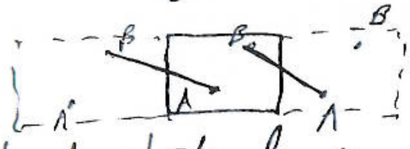


One line that connects A to B is just what you'd find by placing a straightedge on the flattened cylinder: 

But the left and right edges were originally connected. Think of it like a video game screen on which objects that go off the left ~~on~~ reappear on the right: 

. This is the second line.

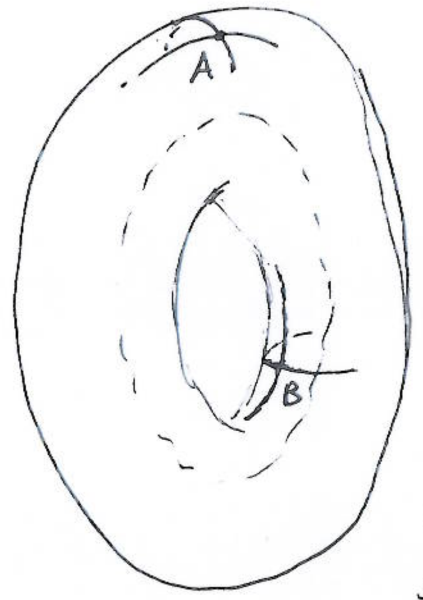
Here's a quick way to find it:



But look at what else is possible: 

… the cylinder any number of times!

2.8

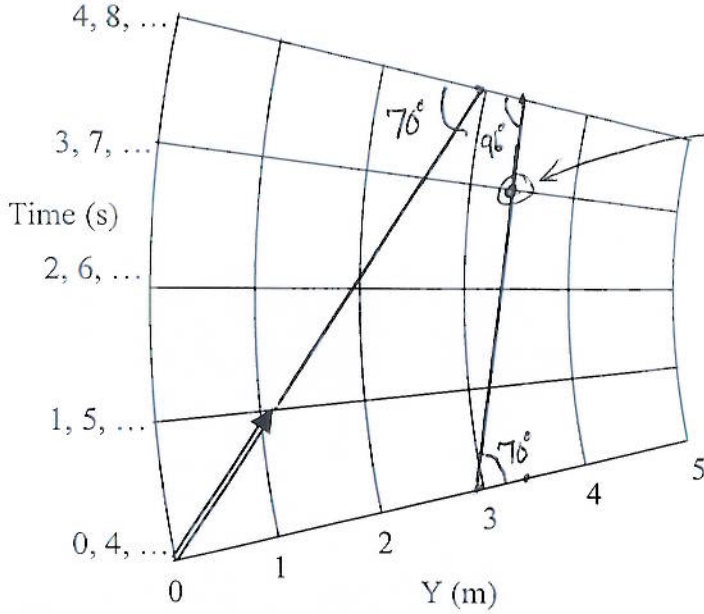


Curvature is positive if the two arcs have their centers on the same side of the surface. At A, this is the case.

At B, the centers are on opposite sides of the surface, so the curvature is negative there.

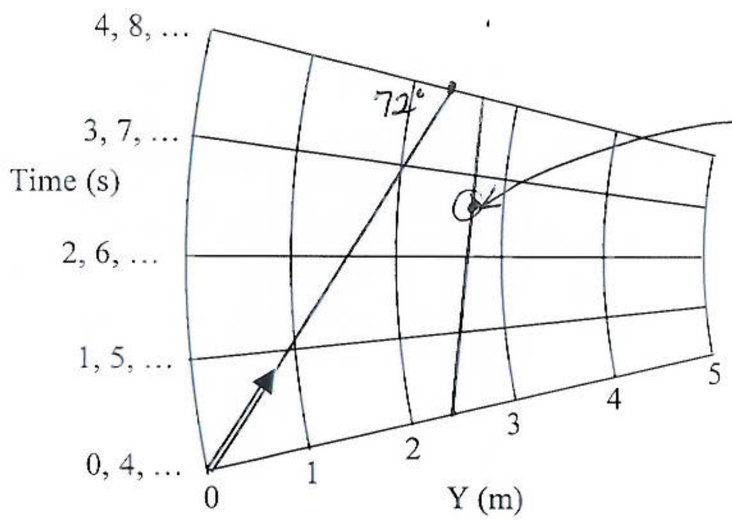
Roughly, points on the inner part of the ~~inner~~ torus have negative curvature while those on the outer region have positive curvature. The boundary between these regions has zero curvature.

3.1



Maximum height = ~3.4m

A



Maximum height = ~2.7m

B

Figure B has more distortion, and the projectile maxed out at a lower height. (More distortion = stronger gravity.)

3.2

More curvature means more distortion of the ~~grap~~ grids seen in ~~fig.~~ problem 3.1. More distortion means the path reaches its maximum y coordinate in less time.

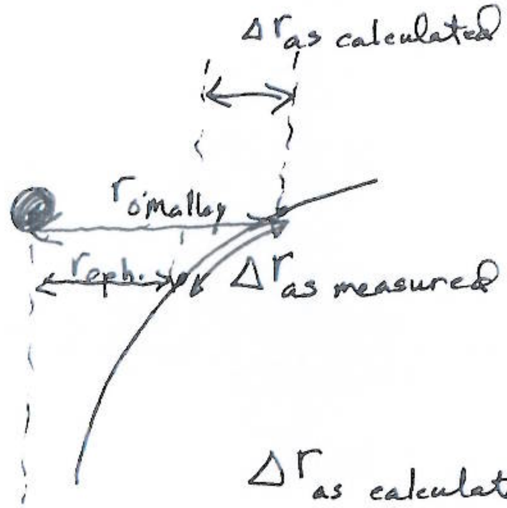
If the curvature were infinite, the straight line path would reach its maximum y coordinate immediately: the projectile, no matter what its speed, could not rise. (I believe there is a connection to a black hole here!)

3.3

The sheet of spandex represents a flat plane of our universe. To "jump off the surface" would be to leave that flat plane. That would increase the distance - not a short cut.

The conceptual hassle here is that analogies have their limitations. The distorted spandex is great at showing how the original plane has been stretched - there are more square centimeters of area when you place a weight on it. But it implies that this is done by bending that plane into a third dimension, and that is <u>not</u> the case. There's just more space.

3.4



$$\Delta r_{\text{as calculated}} = r' - r = 150{,}000\,\text{km} - 100{,}000\,\text{km}$$

$$= 50{,}000\ \text{km}.$$

$\Delta r_{\text{as measured}}$ is larger than this, due to the stretching of space.